Bayesian Updating with Local Varying Correlation

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The Bayesian Updating technique has been successfully applied for many 2-D reservoir modeling projects. The results provide very useful information for delineation planing, reserves estimation and well placement. The capability of integrating multiple variables of different types and easy implementation to model 20 or 30 variables continues to draw attention. Based on practical application experience, the technique can be improved to use Local Varying Correlation (LVC) and explicitly account for the precision of secondary data. LVC is proven to be legitimate in likelihood calculation provided that the same weights are used for each correlation coefficient. Secondary data may come from sources with different quality or from geologic gridding with locally varying quality. The precision of the secondary data can also be accounted for to improve the local estimation. Local Varying Quality (LVQ) is developed and applied for likelihood calculation. The enhanced technique is explained and examples are presented.

Introduction

The advantage of 2-D mapping with Bayesian Updating is the straightforward treatment of many variables from different sources. The influence of the primary information and the secondary information is considered separately. There is no problem to model 20 or more reservoir parameters without the burden of cross variogram inference. The mapping results provide useful information for delineation planning, reserves estimation and well placement.

Conventional geostatistical 2-D mapping is done by kriging the well data to interpolate between the well locations. Local uncertainty in the estimates is given by the kriging variance, which accounts for the closeness and redundancy of the well data. However, the sparse well data are not sufficient to provide a quantitative measure in the interwell regions. It is necessary to integrate secondary information, such as seismic data, dynamic data and geological interpretation, to improve the 2-D modeling. Cokriging, in particular collocated cokriging (Xu *et al.*, 1992), are common geostatistical methods for integration of different types of data; however, inference of the cross-covariance model(s) is demanding from the perspective of professional effort and computational time.

Recently, the Bayesian Updating technique was introduced for data integration. The technique decomposes the collocated cokriging estimate into a production of two models: prior and likelihood models. The prior model is built from primary information, and the likelihood model is built from secondary information. The prior model is then updated with the likelihood model to build the final model. Doyen *et al.*'s original presentation allowed for updating with only one secondary variable at a time. Deutsch and Zanon (2004) proposed a similar approach that also allows multiple secondary variables simultaneously integrated into the mapping of primary variable. This approach has the advantage of easy implementation of mapping of multiple reservoir parameters using multiple secondary variables. It has been successfully applied in the Fort McMurry formation (Ren *et al.*, 2005). The detailed methodology of the Bayesian updating technique is given below.

Bayesian Updating

Suppose a random function Z($\mathbf{u}_i$) is the primary data available at *n* locations in the area of interest where **u** is the location vector and *i=1, …, n*. There are *m* random functions $S_j(\mathbf{u})$ *j=1, …, m* are secondary data available at all locations in the entire model area. After the normal score transform (Deutsch and Journel, 1998), the Y($\mathbf{u}_i$) is the normal scores of the primary data Z($\mathbf{u}_i$), and the $X_j(\mathbf{u})$ is normal scores of secondary data $S_j(\mathbf{u})$. In the context of Bayesian statistical analysis, the results of kriging using only the primary data are considered as a ***prior*** distribution of uncertainty parameterized by:

$$y_P^*(\mathbf{u}) = \sum_{i=1}^{n} \lambda_i y(\mathbf{u}_i) \tag{1}$$

and the weights are calculated from the well known normal equations:

$$\sum_{i=1}^{n} \lambda_i C(\mathbf{u}_i - \mathbf{u}_k) = C(\mathbf{u} - \mathbf{u}_k), \quad k = 1, \ldots n \tag{2}$$

where $C(\mathbf{u}_i - \mathbf{u}_k)$ is the covariance between primary data $y(\mathbf{u}_i)$ and $y(\mathbf{u}_k)$ at distances **h** away, and $C(\mathbf{u} - \mathbf{u}_k)$ is the covariance between estimated location $y(\mathbf{u})$ and primary data $y(\mathbf{u}_k)$ at distances **h** away. The kriging variance is then given by

$$\sigma_{sk}^2(\mathbf{u}) = \sigma^2 - \sum_{i=1}^{n} \lambda_i C(\mathbf{u} - \mathbf{u}_i) \tag{3}$$

Secondary data such as structural information, geological interpretations, and seismic data can be mathematically combined to provide an estimate of the reservoir parameter at each location using a kriging-like equation with the weights calculated from the correlations between different secondary variables and between the secondary variable and primary variable:

$$y_L^*(\mathbf{u}) = \sum_{j=1}^{m} \lambda_j x_j(\mathbf{u}) \tag{4}$$

Here, the weights are also given by the well-known normal equations (cokriging):

$$\sum_{j=1}^{m} \lambda_j \rho_{j,k} = \rho_{j,0}, \quad k = 1, \ldots m \tag{5}$$

where $\rho_{j,k}$ is the correlation between different types of secondary data, and $\rho_{j,0}$ is the correlation between the secondary data and primary data. The likelihood estimation variance is given by:

$$\sigma_L^2(\mathbf{u}) = 1 - \sum_{j=1}^{m} \lambda_j \rho_{j,0} \tag{6}$$

This yields an estimate (y^*_L) and a measure of the secondary variable information content (σ^2_L), forming a distribution of uncertainty that, under this same Bayesian context, is referred to as the ***likelihood***.

The prior information and the likelihood information are then combined to yield the best estimate (with respect to a mean squared error criterion):

$$y_U^* = \frac{y_L^* \sigma_P^2 + y_P^* \sigma_L^2}{(1-\sigma_L^2)(\sigma_P^2 - 1) + 1} \tag{7}$$

and the corresponding variance is calculated as:

$$\sigma_U^2 = \frac{\sigma_P^2 \sigma_L^2}{(1-\sigma_L^2)(\sigma_P^2 - 1) + 1} \tag{8}$$

These results give the parameters of an updated distribution called the ***posterior*** distribution. The mathematics of merging prior and likelihood distributions is well established in statistics; the development of these equations can be found in Doyen *et al.* (1996), and Neufeld and Deutsch (2004). A schematic illustration of the Bayesian Updating technique is given in Figure 1.

Bayesian updating technique relies on the multivariate Gaussian assumption. Under this assumption, all multivariate relationships are linear and can be characterized by correlation coefficients. The correlation coefficient obtained from the wells is considered as true relationship of a pair of variables over the model area and applicable to the interwell regions. However, in practice, the multivariate relationships may not be linear and the correlation coefficients may not be representative of true relationship. There is a risk that the updated estimate is misleaded by a non-representative likelihood. Using multiple secondary variables can reduce the risk. However, a more representative measure of multivariate relationships is more desirable to improve the estimation.

Besides the correlation, the reliability of secondary data can also affect the estimation and the uncertainty assessment. In the Bayesian updating, the secondary data are assumed to be accurate. However, the dense secondary data sometimes do not have same quality over the large model area. The quality of 3-D seismic surveys may be different at different time and different survey conditions. The directly measured values always have better quality than the modeling results that are interpreted from 2-D seismic lines or kriged results. Therefore, the quality of secondary data must be appropriately accounted for.

The Enhanced Bayesian Updating technique will aim to account for the complex multivariate relationships and the quality of secondary data to improve the final estimation and uncertainty assessment.

Some Enhancements

In the Bayesian Updating technique, a representative correlation coefficient is critical for an accurate estimation. The correlation coefficient is calculated using all wells data with the following equation:

$$\rho_{xy} = \frac{\frac{1}{n}\sum_{i=1}^{n} (Z_{xi} - m_x)(Z_{yi} - m_y)}{\sigma_x \cdot \sigma_y} \in [-1, 1] \tag{9}$$

where m_x and m_y are the arithmetic means of variables Z_x and Z_y, respectively. And σ_x and σ_y are the standard deviations of variables Z_x and Z_y, respectively.

The correlation coefficient can only measure the linear relationship between two variables. In most times, the bivariate relationship is not linear. The cross plot is normally used to show the information between two variables. The correlation coefficient can only extract a small part of information in cross plot (Goovaerts, 1997, P21). When using the correlation coefficient, most information is ignored and the linear relationship is assumed to be constant over the whole model area. For example, let's see the correlation between the two variables shown in Figure 2-1. 64 data are extracted from them at locations indicated by black circles. The cross plot of the 64 pairs of data is shown in the right side of Figure 2-1 with a correlation coefficient of 0.533. However, by looking at only 9 data at the up-left and down-right corners, the correlation coefficients are -0.555 and 0.775, respectively (middle column in Figure 2). Of cause, only 9 data may not give the true correlation coefficient. The cross plots of exhaustive data in the two areas are shown in right column in Figure 2. The up-left area actually has a correlation coefficient of 0.156, and the down-right area is 0.636. The local correlations are very different from the global one. Certainly, the relationship of the two variables is much more complex than the linear relationship represented by the global correlation coefficient.

A common problem with correlation coefficient is that it can be significantly affected by the presence of pairs of extreme values. Figure 3 shows that two data make a big different (from 0.590 to 0.728) in the correlation coefficient. The pairs of extreme values are possibly caused by local geological features or error in the measurement. If it is local geological features, neither correlation coefficient can be representative for the whole model area.

Since the local correlation can be much different from the global correlation and the global correlation coefficient is insufficient to descript complex bivariate relationships, the local correlation coefficient should be used for description of bivariate relationships. It is reasonable to assume the local relationship is linear rather than the global relationship. Description of the relationship locally makes it possible to capture the non-linear relationship between any two variables. The more representative correlation is named *locally varying correlation* (LVC).

Locally varying correlation

To calculate local correlation coefficient, it is important to have sufficient data pairs for a representative correlation. Using the moving window method with a large window size will have sufficient data pairs. However, the artifacts caused by the searching windows distort true local correlations. To avoid the artifacts, the weighted correlation coefficient is used. The kriging weights were attempted to use for weighting the nearby data pairs in the calculation of local correlations. However, the map of calculated locally varying correlation does not smooth. In addition, the screening effect in kriging discounts the data pairs behind closer data pairs. Every nearby data pair should be accounted for, and they should be weighted basing on the distance between data and estimation locations. Therefore, the inverse distance method is an appropriate method for weighting data pairs. When using the inverse distance method, the global search can be applied to use all the data to avoid the possible artifacts caused by searching windows.

The inverse distance weights can be calculated by the equation below:

$$w_i = \frac{\dfrac{1}{(d_i + c)^p}}{\displaystyle\sum_{i=1}^{n} \dfrac{1}{(d_i + c)^p}} \tag{10}$$

where *d* is the distance between a datum and the estimated location. And c is an offset value that used to avoid computational problem when estimating at data locations. And *p* is the power of the distance.

Suppose there are *n* pairs of data for random variables *x* and *y* in the domain, at each location, the local weighted mean and variance of *x* are given by:

$$\begin{aligned} m_x &= \sum_{i=1}^{n} w_i Z_{xi} \qquad & i = 1, n \\ \sigma_x^2 &= \sum_{i=1}^{n} w_i Z_{xi}^2 - m_x^2 \qquad & i = 1, n \end{aligned} \tag{11}$$

It is same for variable *y*. The local correlation coefficient is then calculated by

$$\rho_{xy} = \frac{\displaystyle\sum_{i=1}^{n} w_i Z_{xi} Z_{yi} - \sum_{i=1}^{n} w_i Z_{xi} \sum_{i=1}^{n} w_i Z_{yi}}{\sigma_x \cdot \sigma_y} \tag{12}$$

Figure 4 shows the locally varying correlation map for these two variables shown in Figure 1. The local correlations are low at the up-left corner and high at the down-right corner. This is consistent with the local correlations in Figure 2.

The locally varying correlation map for the case of two pairs of extreme values (Figure 3) is shown in Figure 5. The maps of the two variables are shown on the left of the Figure 5. The two pairs are caused by a channel in the secondary data. The locally varying correlation captures the two data. It also shows that the correlation is high in the up-left of the middle area which is a low value area in both maps.

The locally varying correlations are highly dependent on the weights generated from the inverse distance method. The weights, in turn, are dependent on the values used for the power *p* and the offset *c* in Equation10. Firstly, consider the offset *c* is fixed. As the power increases, the weights become more dissimilar, the data far away will have less weight (Isaaks, E. H. and R. M. Srivastava, 1989, P258), and the variation of local correlations will increase. As the power decreases, the weights are more similar, and the variation of local correlations becomes smaller. When the power reaches zero, the local correlations become the global correlation. The changes in weights and variation of local correlations are contrary for the changes of the offset *c*. Now consider the power *p* is fixed. As the offset increases, the weights are more similar, and the variation of the local correlations is smaller. The top two rows of Figure 6 show that the local varying correlations maps become smoother by using larger offset. But the ranges of local varying correlation are smaller. If keeping the same range, we can always increase power and offset to make a smooth local varying correlation map as shown in Figure 6. The difference is that the local varying correlation generated with very small *p* can not reach the full range, and the local varying correlation with very large *p* will look even smoother.

Data impact on the local correlation

The outliers or data pairs of extreme values can dramatically affect the correlation between two variables. Although those outliers can be well captured in the local correlations, it is necessary to identify those data pairs and treated with care. These outliers have a big impact on the local correlation calculation. To identify them, a measure of the impact of each data pair on the local correlation is introduced here. This measure is named *Relative Correlation Difference* (RCD).

The calculation of the RCD is based on a jack knife type method. Firstly, a base LVC map is generated using all data pairs. Then, a data pair is taken out and the rest of data pairs are used to generate a new LVC map. The difference between the two LVC is averaged over the entire model area, and standardized by dividing the base local correlation:

$$RCD = \frac{1}{n}\sum_{i=1}^{n}\frac{(\rho_{Bi} - \rho_{JKi})}{\rho_{Bi}} \quad \in \left[-1,1\right] \tag{13}$$

where the ρ_B is the base local correlation, the ρ_{JK} is the new local correlation, and the n is the number of cells in the model area. The calculated RCD for the data pair is between -1 and 1. The negative sign means the data pair has an impact to lower the local correlation. And the positive sign means an impact to raise the local correlation. Repeat this for another data pair at a time until the RCD for each data pair is calculated. Showing them with kriging map can easily identify the data pairs with large impact.

Figure 7 shows the histogram and kriged map of the RCD of the 64 data pairs of the two variables in Figure 1. The LVC map is also given in the right side of the Figure 7. Note that the RCD is very small with a maximum of 0.07. The eight data pairs with negative RCD make the up-left area have a low correlation. And the two data pairs with positive RCD contribute for the highly correlated down-right area. The RCD also changes with different p and c values. Therefore, it is important to keep the same p and c in the calculation of LVC and RCD.

Bayesian Updating with LVC

Apply the locally varying correlation in Bayesian updating technique is straightforward. It is only used in the likelihood calculation. The global correlation coefficient is replaced with the LVC in Equation 5 and 6. The new matrix of weighted correlations must be positive definite to ensure the likelihood variance is positive and an unique solution to the weights in Equation 5. The positive definiteness of the new weighted correlation matrix has been proved by Olena (see appendix). With the new likelihood results, the prior is updated using same Equations 7 and 8.

The two variables in Figure 1 are used to perform the Bayesian updating and the Enhanced Bayesian updating techniques. The primary variable is used as a reference of true values. The 64 extracted data are used as primary data, and the exhaustive data of secondary variable are used as secondary data. The results of the Bayesian updating and the Enhanced Bayesian updating techniques are given in Figures 8 and 9. The left column shows the reference and the prior and updated models for visual comparison. The kriging results are very smooth. The updated results are much closer to the reference than the kriging results. The right column shows the secondary data, likelihood model and the global or local correlation coefficient model. The likelihood model is generated basing on the local correlation between the primary and secondary data. Where the local correlation is higher, the likelihood model more look like the secondary data, so is the updated model. Therefore, using highly correlated secondary data will generate better updated model. However, using single secondary data that is 100% correlated with the primary data, the

Equations 7 and 8 will not be able to provide correct updated results at well locations. Actually in this case, the secondary data can be directly used as the primary variable.

The Bayesian updated results and the Enhanced Bayesian updated results look very similar. The maps of difference between each updated result and the reference are shown in the first row in Figure 10. The large differences are in the up left area where the local correlations are low. In the down right area, the differences are relatively smaller. The map of difference between the two updated results is also given in the second row in Figure 10. The differences between the two updated results are very small, only about one tenth of the differences between the updated results and the reference. As expected, the difference is mainly in the up-left and down right areas. What are not expected are those areas with positive (yellow or red) and negative (blue) differences have opposite difference between the Bayesian updated results and the reference. This means the EBU estimates in those areas are actually further away from the true values than the BU estimates. Let's see the reason behind this.

In the up-left area, the primary and secondary data are very different with a very low correlation. In the EBU, a low correlation gives a low weight (Equation 5), and a high likelihood variance (Equation 6). The likelihood variance is very sensitive to the low correlation because it is calculated from both the low correlation and the low weight. A high likelihood variance will results in that the updated estimate more close to the prior estimate (Equation 7). Kriging estimates are smooth so that the dramatic changes between data locations are not captured. BU results are not good in capturing them neither because the secondary data in this area does not helpful. This is the place where we need additional secondary data. If there are some secondary data is highly correlated with the primary data in this area, the EBU would perform better than BU. BU gives more weight to the secondary data that has a high global correlation, while the EBU could combine different secondary data for different areas basing on the local correlations.

In the down-right area, there are improvement and misleading for EBU. The misleading is mainly caused by the difference between the primary and secondary data. These differences are located in the interwell regions. Therefore, the correlation would not be able to capture them. This misleading may be reduced by using additional secondary data. The important thing is to have highly correlated, reliable secondary data. The secondary data with poor quality could cause more misleading. This is why the quality of the secondary data should also be considered.

Cross validation at the 64 data location is performed with the two techniques. The results are given in Figure 11. The EBU is more fit with true values at the data locations than the BU. The mean and standard deviation of EBU estimates is more close to the mean and standard derivation of true values, and the correlation between the true values and the updated estimates is also improved. However, by checking with all of the reference values (Figure 12), the BU is better than EBU, which is consistent with what has been seen in Figure 10. This can be changed by using multiple secondary variables.

The quality of secondary data is very important. In the Bayesian Updating technique, the secondary data are assumed to be accurate. In practice, it may not be true. Explicitly accounting for the quality of secondary data can improve the estimation and uncertainty assessment. The *locally varying quality* (LVQ) is introduced for completion of the EBU technique.

Local Varying Quality

The quality of secondary data must be accounted for to improve the 2-D modeling results. The dense secondary data may come from different 3-D seismic surveys or from kriged 2-D seismic lines or well data. Different 3-D seismic surveys may have different qualities depending on different survey techniques and conditions. 2-D seismic data usually has less quality than 3-D seismic data. 2-D seismic lines should have better quality than the estimated locations. Treating them all the same could mislead the final updated estimate and local uncertainty results. The quality of secondary data must be appropriately accounted for.

The locally varying quality defines the percentage of our confident in the data. 3-D seismic data can be assumed to have same quality in the same survey area. The different qualities for different 3-D surveys need to be defined basing on the expert's knowledge. When the secondary data is generated from a kriging-type method, the local varying quality can be calculated using kriging variance as below:

$$Q = C \cdot \sqrt{1-\sigma_k^2} \qquad \in [0,1] \tag{15}$$

where the C is the maximum quality, which is the quality at 2-D seismic lines or well locations. If the secondary data is kriged with well data, the C is 1. The σ_k^2 is the kriging variance, which is the error variance of kriging estimates. Figure 13 shows the curve of data quality verse kriging variance. At well locations, the kriging variance is zero and the quality is 1. As the kriging variance increases, the data quality drops. This situation is very close to the kriging estimation with one datum. The simple kriging system gives the following relationship between the kriging variance σ_k and the correlation ρ between the datum and the estimate:

$$\sigma_k^2 = 1 - \rho^2 \tag{14}$$

If the variance is low, the correlation is high, and the estimate is more close to the true datum. The data quality likes the correlation. If the variance is low, the quality will be high so that the data will be more close to the true values.

Bayesian Updating with LVC and LVQ

The local varying quality can not be used to get true values by multiplying the data. It rather affects the correlation between the secondary data and primary data. Usually, when the quality of secondary data is poor, the data will be more scatted so that the correlation between the secondary and primary data will drop. Therefore, an effective correlation between secondary and primary data can be used. It is calculated by multiplying the LVQ and the LVC:

$$\hat{\rho}_{i,0} = Q \cdot \rho_{i,0} \tag{15}$$

The effective correlation can only apply in likelihood calculation. The likelihood equations can be modified to use the effective correlations:

$$\sum_{j=1}^{n} \lambda_j \cdot \rho_{i,j} = \hat{\rho}_{i,0} \quad i=1,\ldots,n \tag{16}$$

The correlations between different secondary data ρ_{ij} are not changed by assuming the correlations calculated from well data are applicable to the estimated location. The new weights calculated from the likelihood system account for the quality of secondary data. So is the likelihood variance:

$$\sigma_L^2 = 1 - \sum_{i=1}^{n} \lambda_i \cdot \hat{\rho}_{i,0} \tag{17}$$

Because the variance is calculated using the weight and the effective correlation, it will increase fast if the data quality is low.

The Bayesian updating and enhanced Bayesian updating techniques are performed again for the two variables and additional secondary variable, seismic amplitude. Suppose a 3-D seismic survey is conducted for the up-left area, and the amplitude is highly correlated with the primary variable. 2-D seismic is conducted for the rest of model area. Kriging is performed with the 2-D seismic lines to generated amplitude model, and then combined with the 3-D seismic results. The final map of amplitude is given in Figure 14. The associated LVQ is also given in Figure 14. The quality of 3-D seismic area is 1. The quality of 2-D seismic lines is 0.9. The kriged estimates have different qualities from 0.6 to 0.9. The global correlation coefficients of amplitude to primary and secondary variables are 0.593 and 0.483, respectively (Figure 15). The locally varying correlations are also given in Figure 15. The locally varying correlation of amplitude to primary variable is high in the up-left area and low in the rest of area, especially in the up-right corner.

The results of Bayesian updating and Enhanced Bayesian updating techniques are given in Figures 16 and 17. It is same that the left column shows the reference and the prior and updated models for visual comparison. And the right column shows the secondary data, amplitude data, and likelihood model. The likelihood model of enhanced Bayesian updating is better fitted with high and low correlated areas. The updated map of EBU is more close to the reference.

The maps of difference between each updated result and the reference are shown in the first row in Figure 18. The differences are much smaller than the differences when using single secondary variable (Figure 10). The map of difference between the two updated results is also given. The big differences in the up-left area are the improvements in EBU estimates because of the high local correlation. The up-right corner is also improved by the low local correlation used in EBU. The map of difference between the EBU and previous EBU results indicate the significant improvement by using the high correlated and reliable secondary data.

Both cross validation results (Figure 19) and the cross plots of exhaustive estimates verse the reference (Figure 20) indicate that the EBU is performing better than the BU. The correlation between the true values and the updated estimates is improved.

Conclusion

Bayesian updating provides useful estimation and uncertainty assessment in presence of many secondary data. This technique can be enhanced by using local varying correlation and local varying quality. The local varying correlation can capture the local variation and complex relationship between two variables. Using the local varying correlation can improve the integration of secondary information. The quality of secondary data is also important for accurate estimation and uncertainty assessment in the Bayesian Updating technique.

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Appendix

The proof of positive definiteness of LVC matrix established by Olena Babak.

Let $X_1, \dots, X_K$ are *K* random Functions; let $C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1K} \\ C_{21} & C_{22} & \cdots & C_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ C_{K1} & C_{K2} & \dots & C_{KK} \end{bmatrix}$, where

$$C_{kl} = \sum_{i=1}^{N} \lambda_i X_i^k X_i^l - \sum_{i=1}^{N} \lambda_i X_i^k \sum_{i=1}^{N} \lambda_i X_i^l, \; k,l = 1,\dots,K, \text{ and } \sum_{i=1}^{N} \lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1,\dots,N,$$

be the covariance matrix. Let us prove that *C* is positive semidefinite.

Then for fixed $k,l = 1,\dots,K,$ we have

$$C_{kl} = \sum_{i=1}^{N} \lambda_i X_i^k X_i^l - \sum_{i=1}^{N} \lambda_i X_i^k \sum_{i=1}^{N} \lambda_i X_i^l = (X^k)^T diag(\lambda) X^l - (X^k)^T P X^l = (X^k)^T (diag(\lambda) - P) X^l,$$

where $X^l = (X_1^l, \dots, X_N^l)^T$; $X^k = (X_1^k, \dots, X_N^k)^T$; $\lambda = (\lambda_1, \dots, \lambda_N)^T$; $diag(\lambda)$ is a diagonal matrix with vector λ on the diagonal and matrix *P* is given by

$$P = \lambda\lambda^T = \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \cdots & \lambda_1\lambda_n \\ \lambda_1\lambda_2 & \lambda_2^2 & \cdots & \lambda_2\lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1\lambda_n & \lambda_2\lambda_n & \cdots & \lambda_n^2 \end{bmatrix}.$$

And, thus, the covariance matrix C can be rewritten as

$$C = X^T (diag(\lambda) - \lambda\lambda^T) X,$$

where $X = (X^1, \dots, X^K)$.

To prove that C is positive semi-definite, we can prove that matrix $diag(\lambda)-\lambda\lambda^T$ is positive semidefinite, then positive semidefiniteness of C will follow immediately (this is a well known property of the positive semidefinite matrices.

If A is positive semidefinite matrix, so is $X^T AX$, where X is arbitrary matrix.

(This property follows directly from the definition of the positive semidefinite matrices).

Fortunately for us, it is known that matrix $diag(\lambda)-\lambda\lambda^T$ is positive semidefinite matrix (if $\sum_{i=1}^{N}\lambda_i = 1, \quad \lambda_i \geq 0, \quad i = 1,\ldots,N$), it is up to a positive scalar a licit covariance matrix for the multinomial model (see McCullagh and Nelder, 'Generalized linear models', 1989, or http://en.wikipedia.org/wiki/Multinomial_distribution). It can be shown also that this matrix is of rank N-1 and the upper N-1 x N-1 submatrix of it is a positive definite matrix. Thus, we have shown that matrix C is positive semidefinite.

Remark: Because Matrix C is positive semidefinite, so is the corresponding correlation matrix, since $\rho = Y^T CY$, where $Y = Y^T = diag(h)$, $h = \left[\frac{1}{\sigma_1}\cdots\frac{1}{\sigma_K}\right]^T$, and $\sigma_l = \sqrt{VarX^l} > 0$, $l = 1,\ldots,K$.

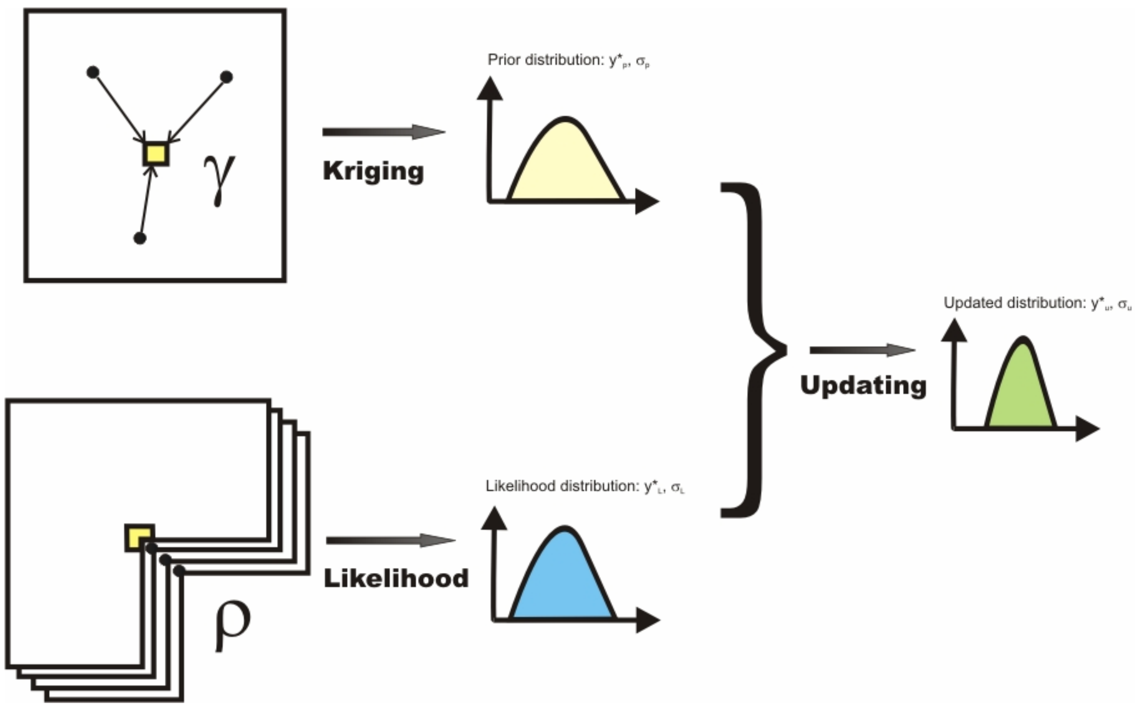

Figure 1: Schematic illustration of the Bayesian Updating technique.

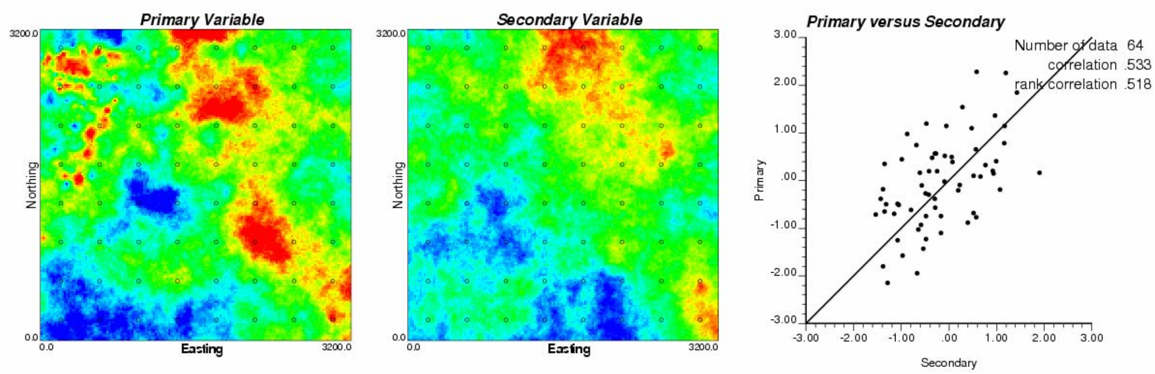

Figure 2: The maps of primary (left) and secondary (middle) variables and their cross plot (right) which indicates a global correlation coefficient of 0.533.

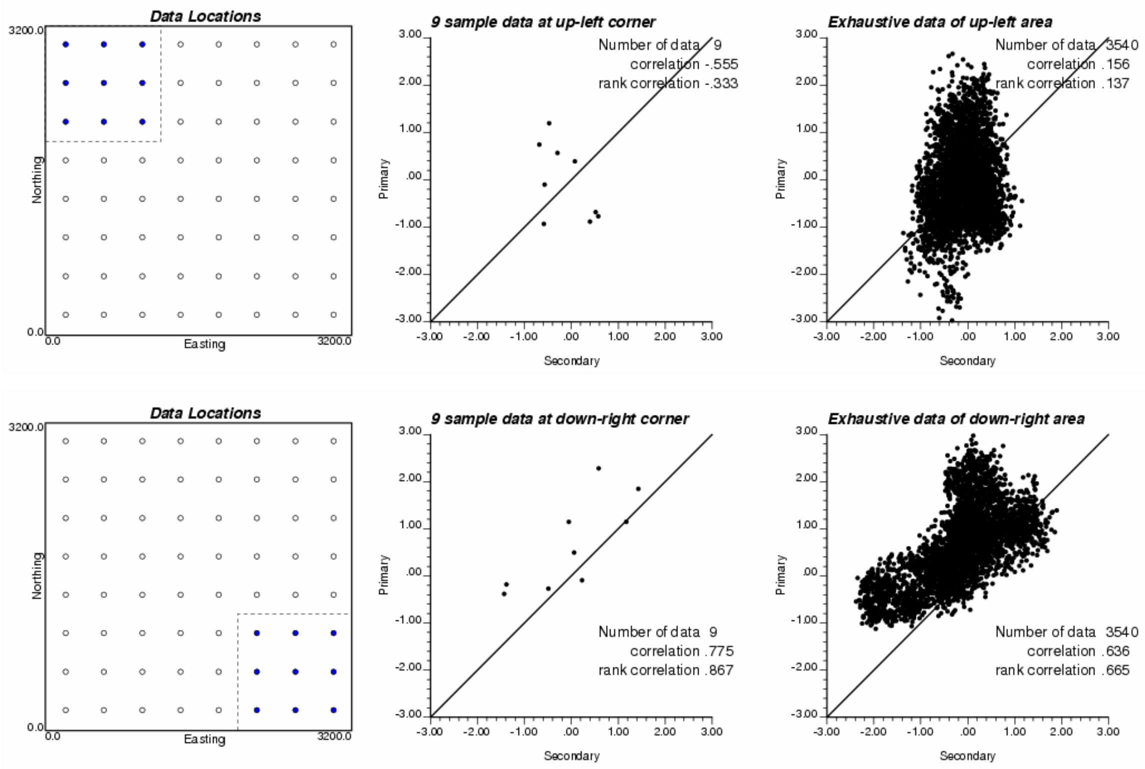

Figure 3: The data locations and the cross plots of the up-left corner (upper row) and the down-right corner (lower row). The middle column shows cross plots of the nine data shown in the location map. The right column shows cross plots of all data in the areas.

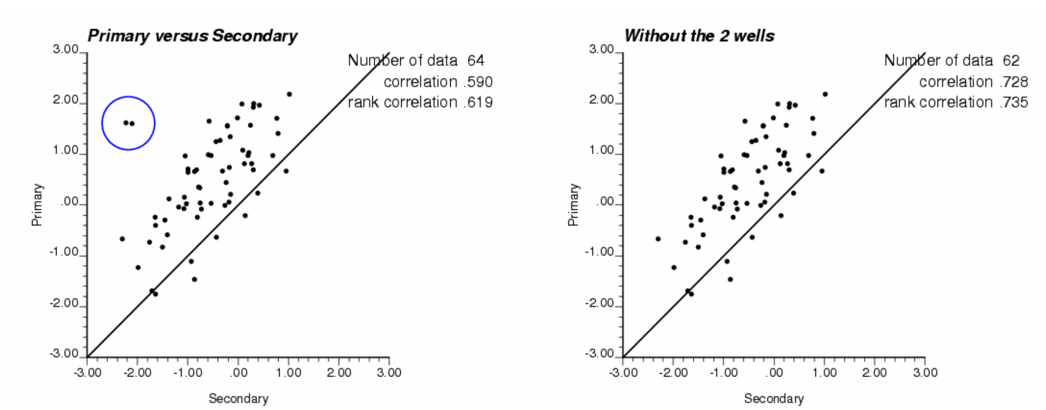

Figure 4: The cross plot of data including the two data pairs in the blue circle gives a correlation of 0.590 (left). The cross plot of data without the two pairs gives a correlation of 0.728 (right).

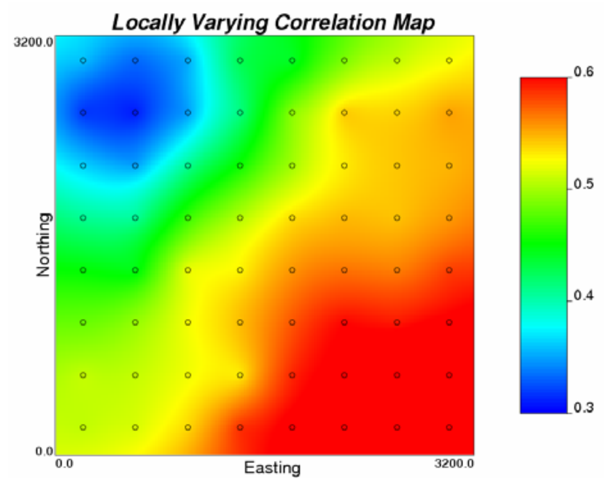

Figure 5: The locally varying correlation map of the two variables shown in Figure 1.

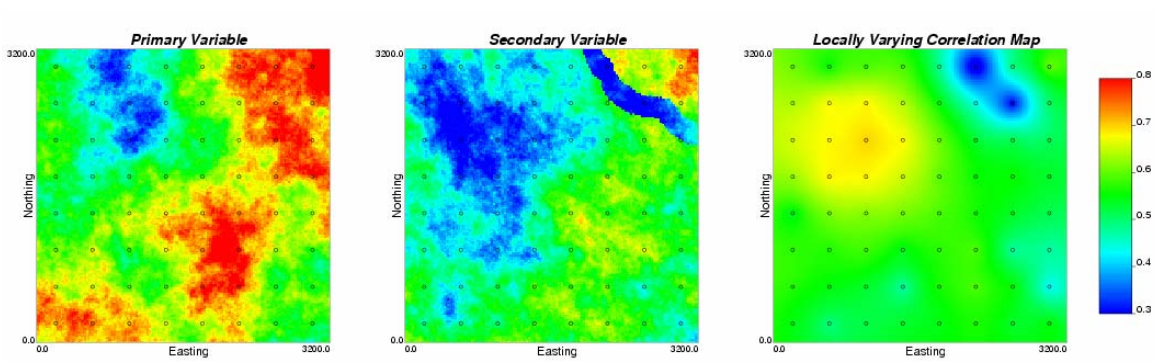

Figure 6: The maps of two variables (left and middle) and their locally varying correlation (right) for the case of two pairs of extreme values shown in Figure 3.

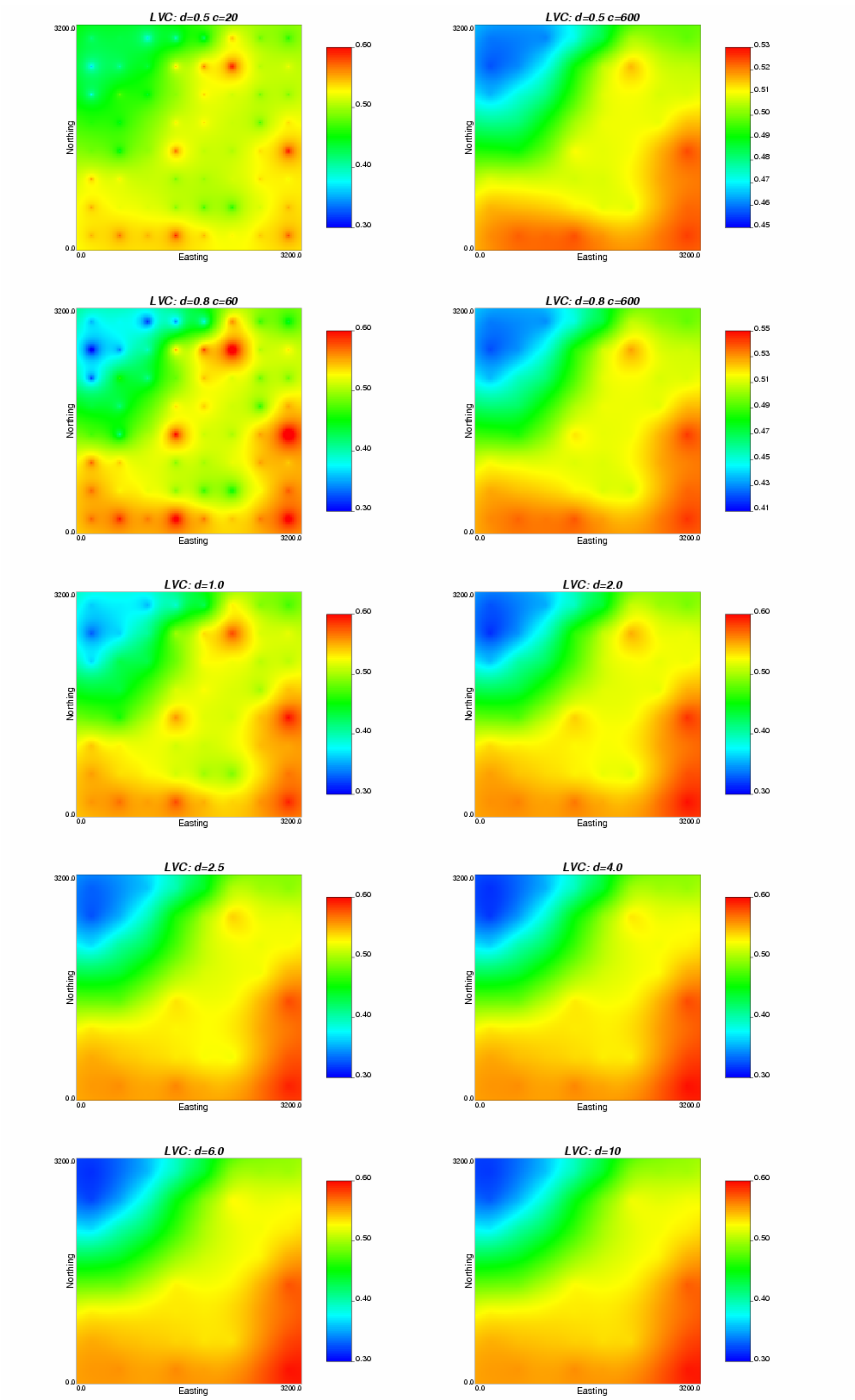

Figure 7: The local varying correlation maps with different p and c for the same value range, except the first two maps in the right column.

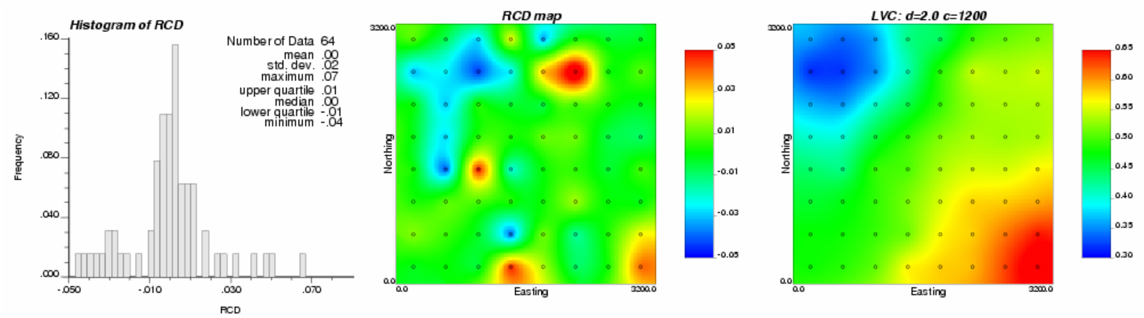

Figure 8: The histogram and kriged map (middle) of RCD and the corresponding LVC map (right) of the two variables shown in Figure 1.

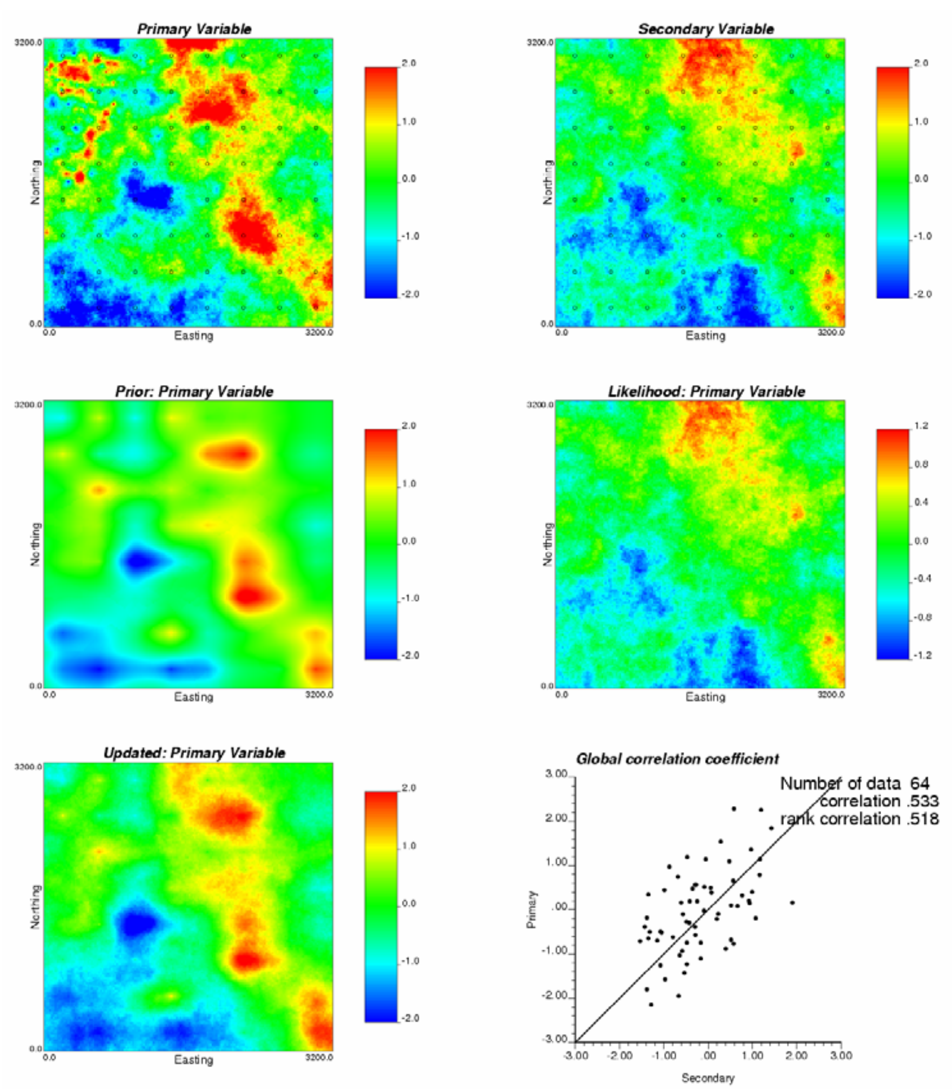

Figure 9: The Bayesian updating results using the global correlation coefficient.

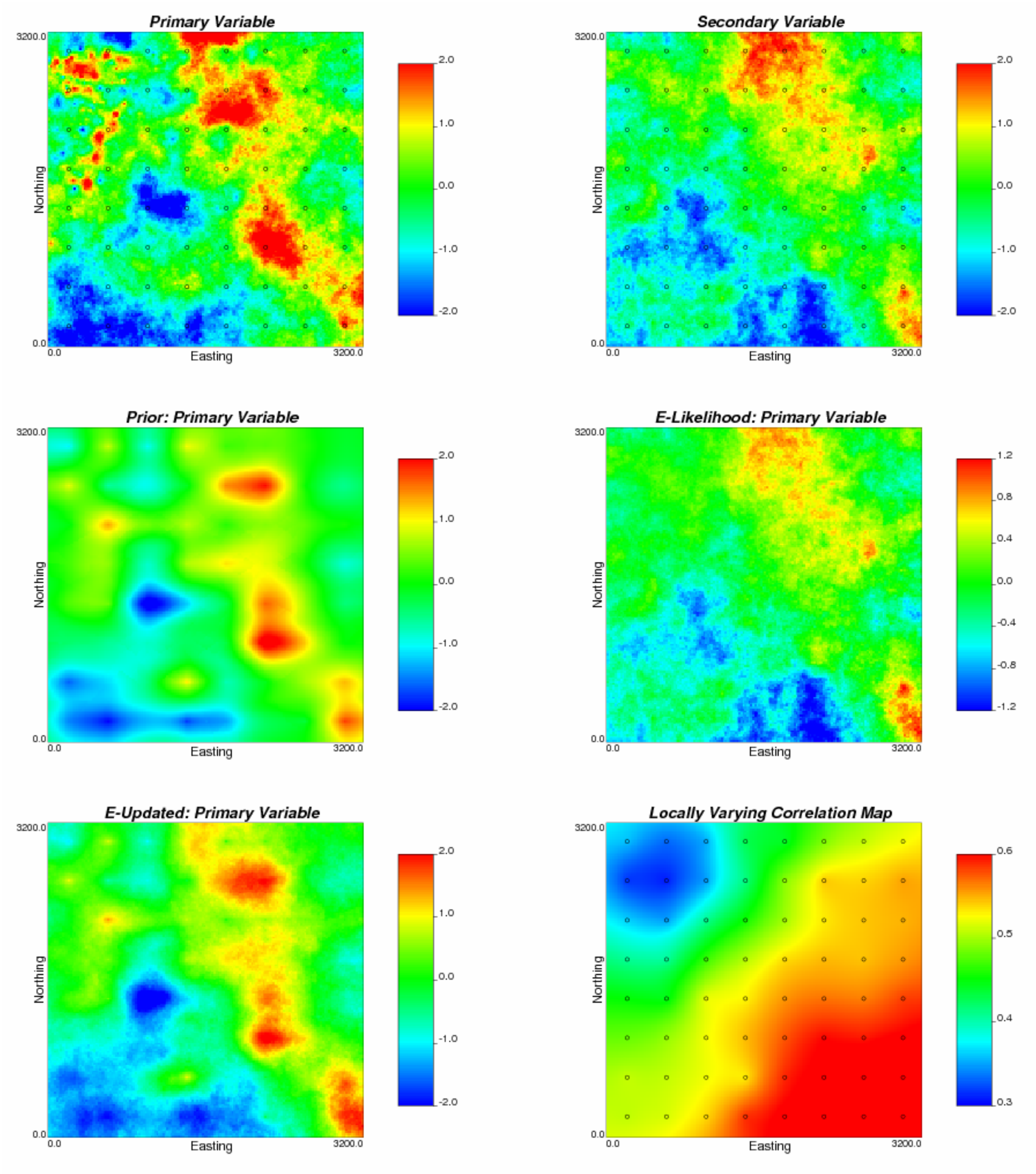

Figure 10: The maps of Bayesian updating with local varying correlations.

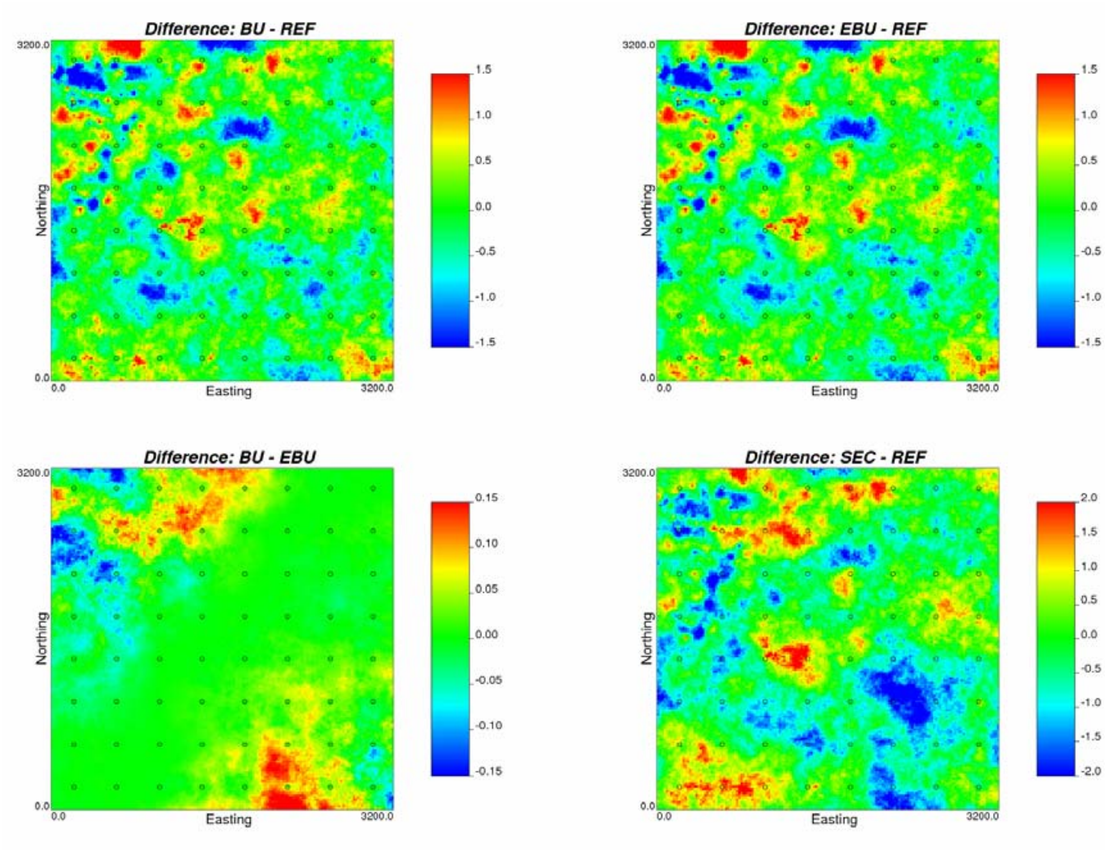

Figure 11: The maps of the difference between the updated results and the reference (first row) and the difference between the Bayesian updating (BU) and Enhanced Bayesian updating (EBU) results (left of secondary row) and the difference between the secondary data and the reference (right of secondary row).

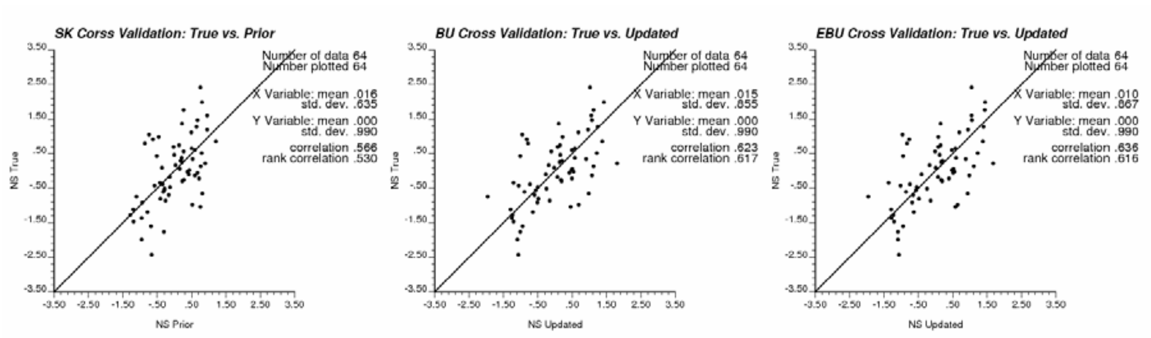

Figure 12: The cross validation of simple kriging (left), Bayesian updating (middle) and enhanced Bayesian updating (right).

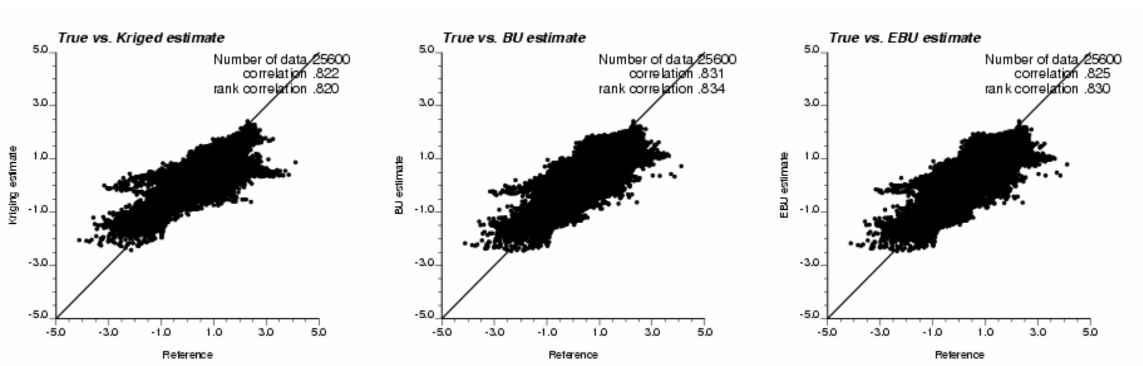

Figure 13: The cross plots of the estimate verses the reference for simple kriging (left), Bayesian updating (middle), and the enhanced Bayesian updating (right).

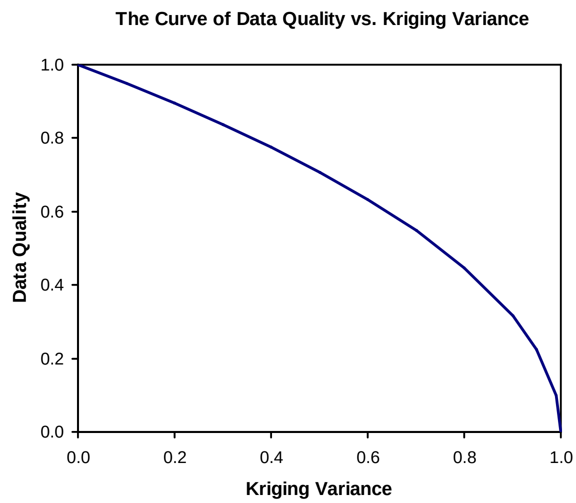

Figure 14: The curve of data quality verse kriging variance with C is 1.

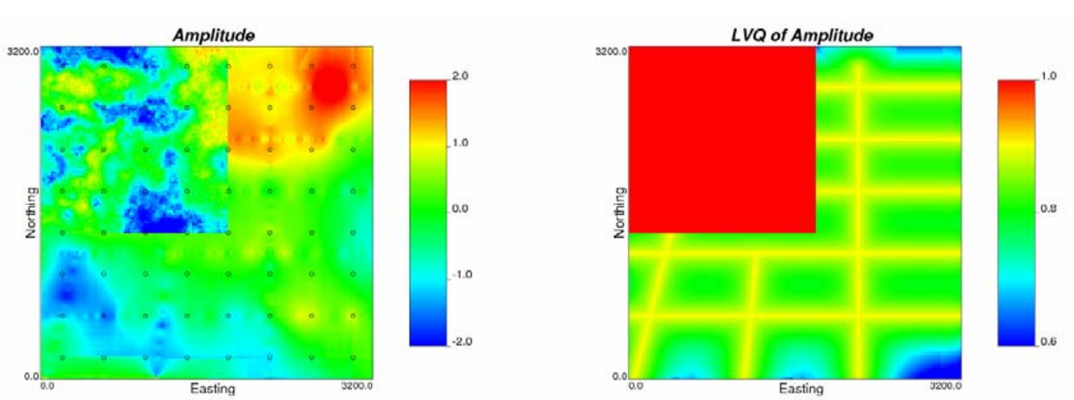

Figure 15: The map of amplitude and local varying quality. The up-left area is the 3-D seismic area, and the rest of the area is modeled by kriging with 2-D seismic lines.

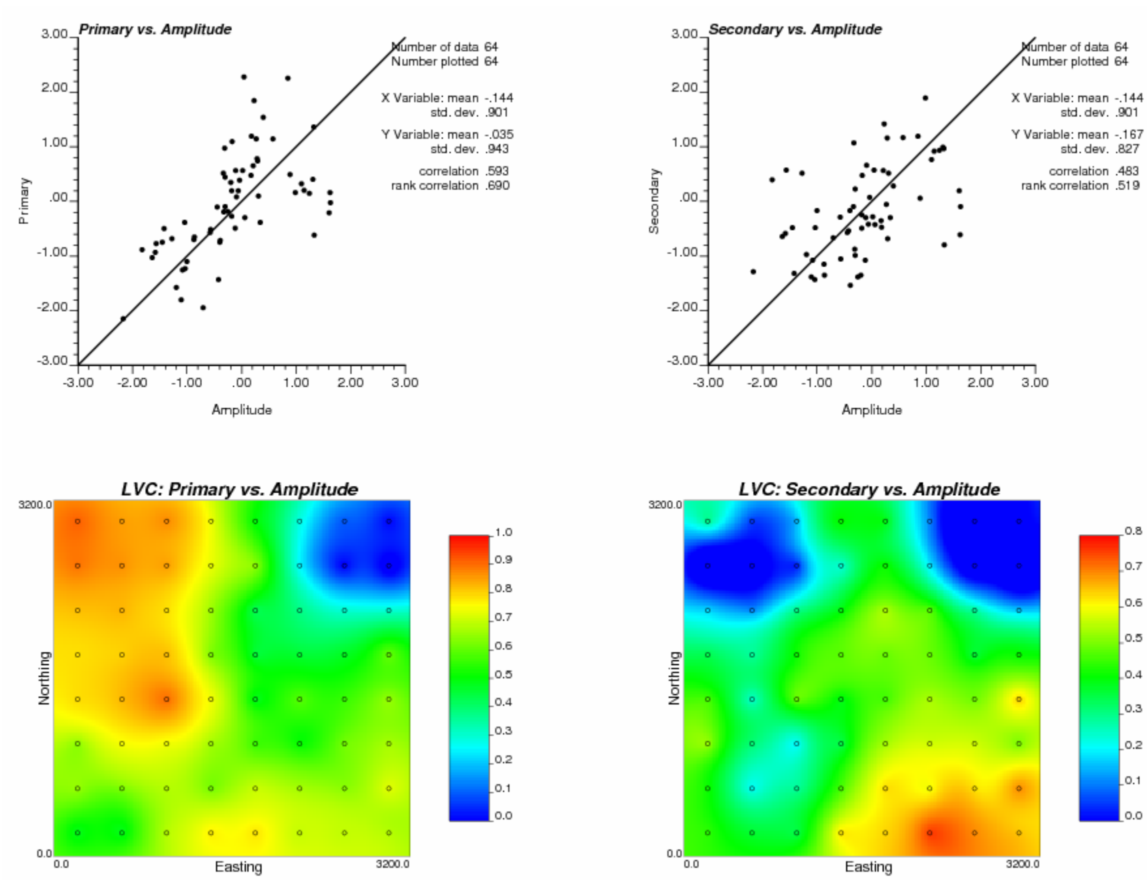

Figure 16: The global correlation and the map of locally varying correlation between primary and amplitude (left column) and between secondary and amplitude (right column).

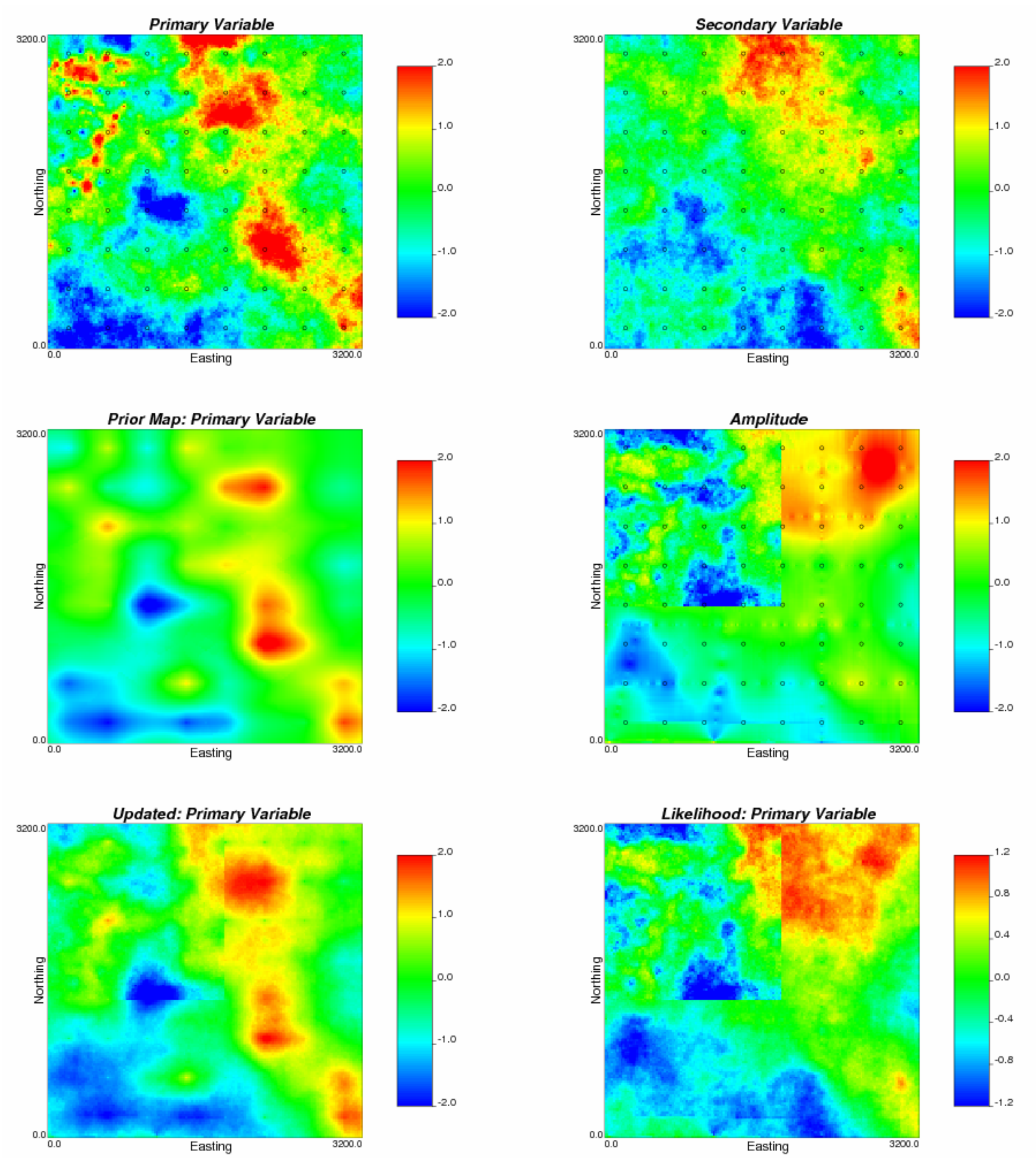

Figure 17: The results of Bayesian updating using two secondary data: Secondary and Amplitude.

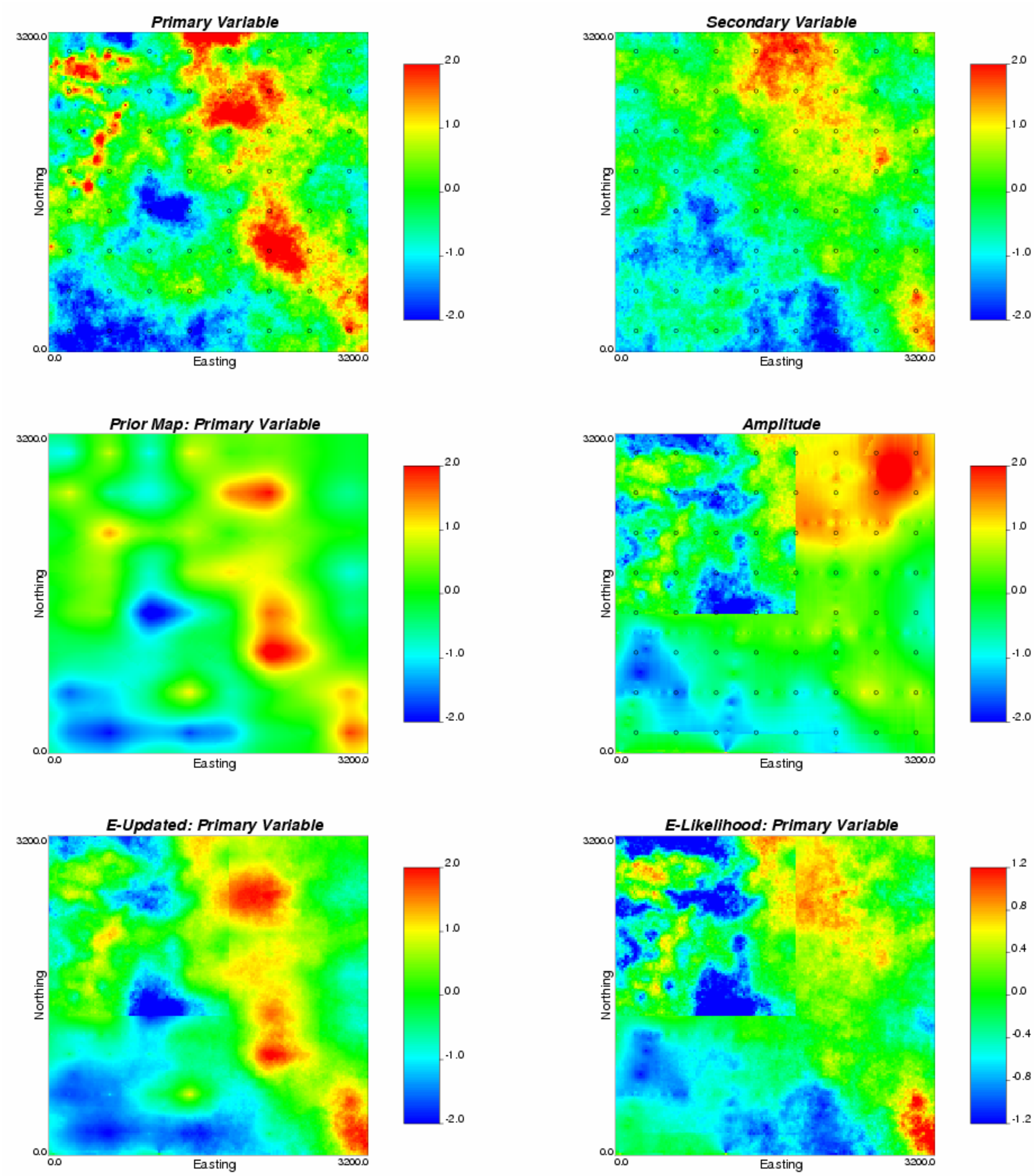

Figure 18: The results of enhanced Bayesian updating using two secondary data: Secondary and Amplitude.

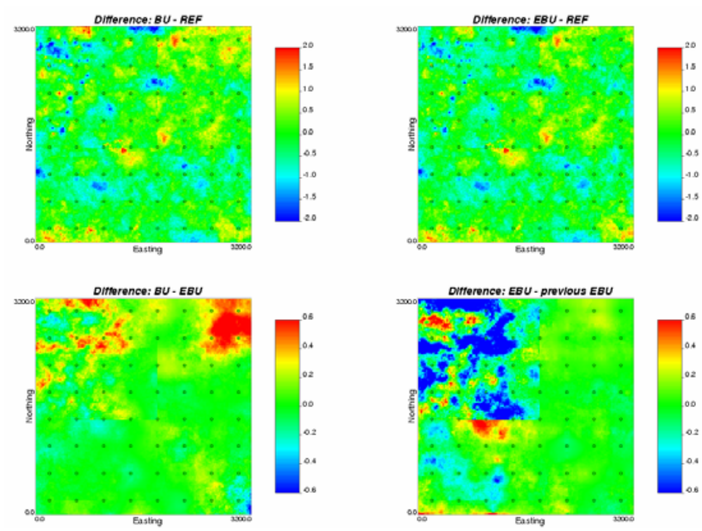

Figure 19: The maps of the difference between the updated results and the reference (first row) and the difference between the Bayesian updating (BU) and Enhanced Bayesian updating (EBU) results (left of secondary row) and the difference between the EBU and the EBU with only one secondary data (right of secondary row).

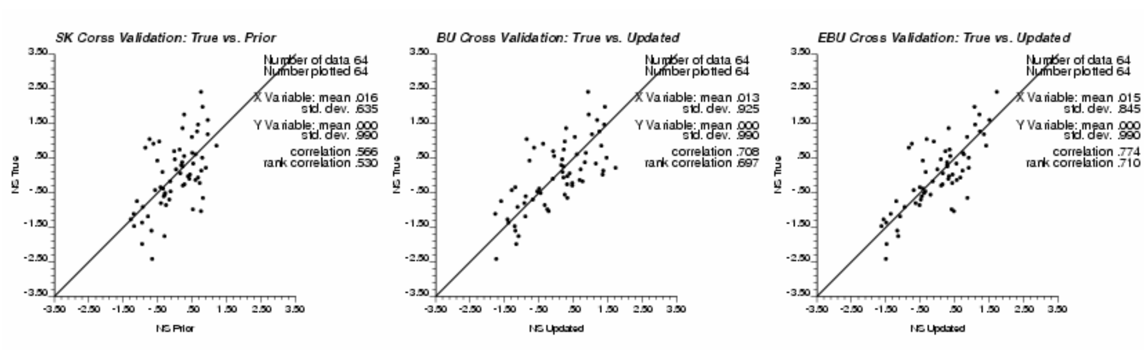

Figure 20: The cross validation of simple kriging (left), Bayesian updating (middle) and enhanced Bayesian updating (right).

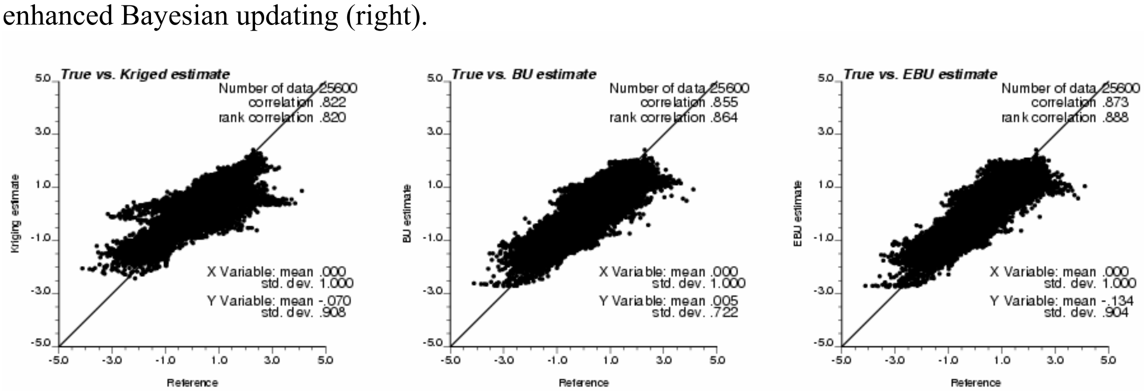

Figure 21: The cross plots of the estimates verses the reference for simple kriging (left), Bayesian updating (middle), and the enhanced Bayesian updating (right).